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A shrinking projection extragradient algorithm for equilibrium problem and fixed point problem

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Abstract

In this paper, a shrinking projection algorithm based on the extragradient iteration method for finding a common element of the set of common fixed points of a finite family of asymptotically nonexpansive mappings and a generalized nonexpansive set-valued mapping and the set of solutions of equilibrium problem for pseudomonotone and Lipschitz-type continuous bifunctions is introduced and investigated in Hilbert spaces. Moreover, the strong convergence of the sequence generated by the proposed algorithm is derived under some suitable assumptions. These results are new and develop some recent results in this field.

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1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is called:

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$,

(ii) quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and $\|Tx - Tp\| \leq \|x - p\|$, for all $x \in C$ and $y \in F(T)$,

(iii) asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in C$ and all $n \geq 1$ we have

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [16] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Moreover, the fixed points set $F(T)$ of T is closed and convex.

A subset $C \subset H$ is called proximal if for each $x \in H$, there exists an element $y \in C$ such that

$$\| x - y \| = dist(x, C) = inf\{\| x - z \| : z \in C\}.$$

We denote by $CB(C), K(C)$ and $P(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C respectively. The Hausdorff metric $\mathfrak{h}$ on $CB(H)$ is defined by

$$\mathfrak{h}(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

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for all $A, B \in CB(H)$.
Let $T : H \to 2^H$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of T, if $x \in Tx$. A multivalued mapping $T : H \to CB(H)$ is called (i) nonexpansive if $\mathfrak{h}(Tx, Ty) \leq \|x - y\|, \quad x, y \in H$,(ii) quasi-nonexpansive if $F(T) \neq \varnothing$ and $\mathfrak{h}(Tx, Tp) \leq \| x - p \|$ for all $x \in H$ and all $p \in F(T)$. Recently, J.Garcia-Falset, E. Llorens-Fuster and T. Suzuki [15], introduced a new generalization of the concept of a nonexpansive single valued mapping which called condition (E). Very recently, Abkar and Eslamian [1], modify the condition (E) for multivalued mappings as follows:

Definition 1.1. A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (E_μ) provided that

$$dist(x, Ty) \leq \mu\, dist(x, Tx) + \|x - y\|, \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Recently, Eslamian and Abkar proved a common fixed point theorem for a commuting pair of mappings, including a finite family of asymptotically nonexpansive mapping and a generalized nonexpansive multivalued mapping in a uniformly convex Banach space (see [2]).
Iterative methods for approximating fixed point points of nonlinear mappings and solutions to variational inequality have recently been studied by many authors. For details, we can refer to [6, 8, 13, 17, 18, 22, 24, 25]. In an infinite dimensional Hilbert space, Mann iteration processes have only weak convergence, in general, even for nonexpansive mappings. In order to obtain a strong convergence theorem for the Mann iterative process to nonexpansive mappings, Nakajo and Takahashi [23], used two closed convex sets that are created in order to form the sequence via metric projection, so that the strong convergence is guaranteed. Later on, it was often referred to as the hybrid algorithm or the CQ method. After that, the hybrid algorithm have been studied extensively by many authors, particularly, Martinez-Yanes and Xu [20], extended some results of Nakajo and Takahashi [23] to the Ishikawa iteration process. Very recently, Takahashi, Takeuchi and Kubota [28] introduced the shrinking projection method which just involved one closed convex set for nonexpansive mappings in a Hilbert space.

Let f be a bifunction from $C \times C$ into $\mathbb{R}$, such that $f(x, x) = 0$ for all $x \in C$. The Equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0, \qquad \forall y \in C.$$

The set of solutions is denoted by $Sol(f, C)$. Such problems arise frequently in mathematics, physics, engineering, game theory, transportation, electricity market, economics and network. Due to importance of the solutions of such problems, many researchers are working in this area and studying on the existence of the solutions of such problems. For example, see; [7, 10, 14]. On the other hand, Tada and Takahashi [26] introduced the CQ method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping T in a Hilbert space H. In recent years, the problem to find a common point of the solution set of equilibrium problem and the set of fixed points of a nonexpansive mapping becomes an attractive field for many researchers (see [9, 12, 21, 26, 27, 28]). We recall the following well-known definitions.
A bifunction $f : C \times C \to \mathbb{R}$ is said to be (i) strongly monotone on C with $\alpha > 0$ iff $f(x, y) + f(y, x) \leq -\alpha\|x - y\|^2, \quad \forall x, y \in C$; (ii) monotone on C iff $f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C$; (iii) psedomonotone on C iff $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \quad \forall x, y \in C$; (iv) Lipschitz-type continuous

on C with constants $c_1 > 0$ and $c_2 > 0$ iff $f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2$, for all $x, y, z \in C$.

If $f(x,y) = \langle Fx, y-x\rangle$ for every $x, y \in C$, where F is a mapping from C into H, then the equilibrium problem becomes the classical variational inequality problem which is formulated as finding a point $x^* \in C$ such that

$$\langle Fx^*, y - x^*\rangle \geq 0, \qquad \forall y \in C.$$

The set of solutions of this problem is denoted by $VI(F,C)$. Recently, P.N. Anh [4, 5], consider the CQ method for finding a common element of the set of solutions of monotone, lipschitz-type continuous equilibrium problem and the set of fixed points of a nonexpansive mapping T in a Hilbert space H.

Theorem 1.2. Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction with constants c_1 and c_2. Suppose that $f(x,.)$ is convex and subdifferentiable on C for all $x \in C$. Let, $T : C \to C$ be a nonexpansive mapping. Assume that $F(T) \bigcap Sol(f,C) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_0 \in C$ and then by

$$\begin{cases} w_n = argmin\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : \quad w \in C\}, \\ u_n = argmin\{\lambda_n f(w_n, u) + \frac{1}{2}\|u - x_n\|^2 : \quad u \in C\}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tu_n, \quad \forall n \geq 0, \\ C_n = \{u \in C : \|y_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_{C_n}} x_0. \end{cases}$$

Assume that the control sequences $\{\alpha_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [a,1) \subset (0,1)$,

(ii) $\{\lambda_n\} \subset [a,b] \subset (0, min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$.

Then the sequences $\{u_n\}$ and $\{x_n\}$ converge strongly to $P_{F(T)\bigcap Sol(f,C)} x_0$.

In this paper, we introduce a shrinking projection algorithm based on the extragradient iteration method for finding a common element of the set of fixed points of a finite family of asymptotically nonexpansive mappings and a generalized nonexpansive multivalued mapping and the set of solutions of equilibrium problems for pseudomonotone and Lipschitz-type continuous bifunctions in a Hilbert space. Moreover, the strong convergence of the sequence generated by the proposed algorithm is derived under some suitable assumptions.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let C be a nonempty closed convex subset of H. Let the symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. Let C be a closed convex subset of a Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C.$$

The mapping P_C is called the metric projection of H onto C.

Lemma 2.1. ([28]) Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, then $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \qquad \forall y \in C.$$

Lemma 2.2. ([23]) Let C be a closed convex subset of H. Then for all $x \in H$ and $y \in C$ we have

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2.$$

Lemma 2.3. [12] Let H be a Hilbert space and $x_i \in H$, $(1 \leq i \leq m)$. Then for any given $\{\lambda_i\}_{i=1}^m \subset]0,1[$ with $\sum_{i=1}^m \lambda_i = 1$ and for any positive integer k, j with $1 \leq k < j \leq m$,

$$\|\sum_{i=1}^m \lambda_i x_i\|^2 \leq \sum_{i=1}^m \lambda_i \|x_i\|^2 - \lambda_k \lambda_j \|x_k - x_j\|^2.$$

Lemma 2.4. [3, 11]Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a quasi-nonexpansive multivalued mapping. If $F(T) \neq \varnothing$, and $T(p) = \{p\}$ for all $p \in F(T)$. Then $F(T)$ is closed and convex.

Lemma 2.5. [5] Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $f : C \times C \to \mathbb{R}$ be a psedumonotone, and Lipschitz-type continuous bifunction. For each $x \in C$, let $f(x, .)$ be convex and subdifferentiable on C. Let $\{x_n\}, \{z_n\}$, and $\{w_n\}$ be sequences generated by $x_0 \in C$ and by

$$\begin{cases} w_n = argmin\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : \quad w \in C\}, \\ z_n = argmin\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : \quad z \in C\}. \end{cases}$$

Then for each $x^\star \in Sol(f, C)$,

$$\|z_n - x^\star\|^2 \leq \|x_n - x^\star\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2, \quad \forall n \geq 0.$$

3 Main result

Now, we are in a position to give our main results.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction. Suppose that $f(x, .)$ is convex and subdifferentiable on C for all $x \in C$. Let, $T : C \to CB(C)$ be a quasi-nonexpansive multivalued mappings satisfying the condition (E) and $S_i : C \to C, (i = 1, 2, ..., m)$, be a finite family of asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $k_n \to 1$, where $k_n = max\{k_{n,i};\ 1 \leq i \leq m\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^m F(S_i) \bigcap F(T) \bigcap Sol(f, C) \neq \varnothing$ and $T(p) = \{p\}$ for each $p \in \mathcal{F}$. For $C_0 = C$, let $\{x_n\}$ be sequence generated initially by an arbitrary element $x_0 \in C$ and then by

$$\begin{cases} w_n = argmin\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : \quad w \in C\}, \\ u_n = argmin\{\lambda_n f(w_n, u) + \frac{1}{2}\|u - x_n\|^2 : \quad u \in C\}, \\ y_n = \alpha_n u_n + \beta_n z_n + \gamma_{n,1} S_1^n u_n + ... + \gamma_{n,m} S_m^n u_n, \quad \forall n \geq 0, \\ C_{n+1} = \{u \in C_n : \|y_n - u\|^2 \leq \|x_n - u\|^2 + (k_n^2 - 1)\eta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases}$$

where $z_n \in Tu_n$, and $\eta_n = \sup\{\|x_n - u\|^2 : u \in \mathcal{F}\} < \infty$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\} \subset [l, 1) \subset (0, 1)$, $\alpha_n + \beta_n + \sum_{i=1}^{m} \gamma_{n,i} = 1$ $(i = 1, 2, \cdots, m)$,

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, where $L = max\{2c_1, 2c_2\}$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof. We observe that C_n is closed and convex, (see [20]). To show that $\mathcal{F} \subset C_n$ for all $n \geq 0$, take $q \in \mathcal{F}$. By Lemma 2.5 we have

$$\|u_n - q\| \leq \|x_n - q\|.$$

Since T is quasi-nonexpansive and $Tq = \{q\}$, we have

$$\|z_n - q\| = dist(z_n, Tq) \leq \mathfrak{h}(Tu_n, Tq) \leq \|u_n - q\|.$$

Also, from Lemma 2.5, we have

$$\|u_n - q\|^2 \leq \|x_n - q\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - u_n\|^2.$$

Now applying Lemma 2.3 and our assumption we have that

$$\begin{aligned}
\|y_n - q\|^2 \quad &= \|\alpha_n u_n + \beta_n z_n + \gamma_{n,1} S_1^n u_n + ... + \gamma_{n,m} S_m^n u_n - q\|^2 \\
&\leq \alpha_n \|u_n - q\|^2 + \beta_n \|z_n - q\|^2 + \gamma_{n,1} \|S_1^n u_n - q\|^2 + ... + \gamma_{n,m} \|S_m^n u_n - q\|^2 \\
&- \alpha_n \beta_n \|z_n - u_n\|^2 - \alpha_n \gamma_{n,i} \|S_i^n u_n - u_n\|^2 \\
&\leq \alpha_n \|x_n - q\|^2 + \gamma_{n,1} k_n^2 \|u_n - q\|^2 + ... + \gamma_{n,m} k_n^2 \|u_n - q\|^2 \\
&- \alpha_n \beta_n \|z_n - u_n\|^2 - \alpha_n \gamma_{n,i} \|S_i^n u_n - u_n\|^2 \\
&- \alpha_n (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - \alpha_n (1 - 2\lambda_n c_2)\|w_n - u_n\|^2 \\
&\leq (1 + (k_n^2 - 1))\|x_n - q\|^2 - \alpha_n \beta_n \|z_n - u_n\|^2 - \alpha_n \gamma_{n,i} \|S_i^n u_n - u_n\|^2 \\
&- \alpha_n (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - \alpha_n (1 - 2\lambda_n c_2)\|w_n - u_n\|^2.
\end{aligned} \tag{1}$$

Therefore $\|y_n - q\|^2 \leq (1 + (k_n^2 - 1))\|x_n - q\|^2$, and hence $q \in C_n$, which implies that

$$\mathcal{F} = \bigcap_{i=1}^{m} F(S_i) \bigcap F(T) \bigcap Sol(f, C) \subset C_n, \qquad \forall n \geq 0.$$

Now we show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Put $w = P_{\mathcal{F}} x_0$ (we note that $\mathcal{F}$ is closed and convex). From $w \in \mathcal{F} \subset C_n$ and $x_n = P_{C_n} x_0$ for all $n \geq 0$, we get

$$\|x_n - x_0\| \leq \|w - x_0\|.$$

Also from $x_n = P_{C_n} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$ we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|.$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Hence $\lim_{n\to\infty}\|x_n - x_0\|$ exists. We show that $\lim_{n\to\infty} x_n = x^* \in C$. For $m > n$ we have $x_m = P_{C_m}x_0 \in C_m \subset C_n$. Now by applying Lemma 2.2 we have

$$\|x_m - x_n\|^2 \le \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, it follows that $\{x_n\}$ is a Cauchy sequence, and hence there exists $x^* \in C$ such that $\lim_{n\to\infty} x_n = x^*$. Putting $m = n+1$, in the above inequality we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2}$$

In view of $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we see that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + (k_n^2 - 1)\eta_n.$$

It follows that $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0$. This implies that $\lim_{n\to\infty} y_n = x^*$. Observing (1) and our assumption, we have

$$l^2\|u_n - z_n\|^2 \le \alpha_n\beta_n\|u_n - z_n\|^2 \le k_n^2\|x_n - q\|^2 - \|y_n - q\|^2.$$

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$ and $\lim_{n\to\infty} k_n = 1$, we obtain that $\lim_{n\to\infty}\|u_n - z_n\| = 0$, thus

$$\lim_{n\to\infty} dist(u_n, Tu_n) \le \lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3}$$

Using a similar method we obtain that

$$\lim_{n\to\infty}\|u_n - S_i^n u_n\| = \lim_{n\to\infty}\|u_n - w_n\| = \lim_{n\to\infty}\|x_n - w_n\| = 0 \tag{4}$$

From (4) and inequality $\|x_n - u_n\| \le \|x_n - w_n\| + \|w_n - u_n\|$ we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{5}$$

Since $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we have $u_n \to x^*$ as $n \to \infty$. Applying (2) and (5) we get

$$\|u_n - u_{n+1}\| \le \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - u_n\| \to 0 \quad as \qquad n \to \infty. \tag{6}$$

From (4) and (6) for each $i \in \{1, 2, ..., m\}$ we have

$$\begin{aligned}\|u_{n+1} - S_i^n u_{n+1}\| &\le \|u_{n+1} - u_n\| + \|u_n - S_i^n u_n\| + \|S_i^n u_n - S_i^n u_{n+1}\| \\ &\le \|u_{n+1} - u_n\| + \|u_n - S_i^n u_n\| + k_n\|u_n - u_{n+1}\| \to 0 \quad as \quad n \to \infty,\end{aligned}$$

hence

$$\begin{aligned}\|u_{n+1} - S_i u_{n+1}\| &\le \|u_{n+1} - S_i^{n+1} u_{n+1}\| + \|S_i^{n+1} u_{n+1} - S_i u_{n+1}\| \\ &\le \|u_{n+1} - S_i^{n+1} u_{n+1}\| + k_1\|S_i^n u_{n+1} - u_{n+1}\| \to 0 \quad as \quad n \to \infty.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|u_n - S_i u_n\| = 0; \qquad (i = 1, 2, \cdots, m). \tag{7}$$

We observe that $x^* \in \bigcap_{i=1}^m F(S_i)$. Indeed,

$$\begin{aligned} \|x^* - S_i x^*\| &\leq \|x^* - u_n\| + \|u_n - S_i u_n\| + \|S_i u_n - S_i x^*\| \\ &\leq (k_1 + 1)\|x^* - u_n\| + \|u_n - S_i u_n\| \to 0 \quad as \quad n \to \infty, \end{aligned}$$

which implies that $x^* = S_i x^*$. Also we have $x^* \in F(T)$. Indeed,

$$\begin{aligned} dist(x^*, Tx^*) &\leq \|x^* - u_n\| + dist(u_n, Tx^*) \\ &\leq 2\|x^* - u_n\| + \mu\, dist(u_n, Tu_n) \to 0 \quad as \quad n \to \infty, \end{aligned}$$

hence $x^* \in F(T)$. Applying (4) and (5) we get that $x^* \in Sol(f, C)$, (for details see [5]). Hence $x^* \in \mathcal{F}$. Now we show that $x^* = P_{\mathcal{F}} x_0$. Since $x_n = P_{C_n} x_0$, by Lemma 2.1 we have

$$\langle z - x_n, x_0 - x_n \rangle \leq 0, \qquad \forall z \in C_n.$$

Since $x^* \in \mathcal{F} \subset C_n$ we get

$$\langle z - x^*, x_0 - x^* \rangle \leq 0, \qquad \forall z \in \mathcal{F}.$$

Now by Lemma 2.1 we obtain that $x^* = P_{\mathcal{F}} x_0$. Q.E.D.

Now we remove the restriction $T(p) = \{p\}$ for all $p \in F(T)$. Let $T : C \to P(C)$ be a multivalued mapping and

$$P_T(x) = \{y \in Tx : \|x - y\| = dist(x, Tx)\}.$$

We have $F(T) = F(P_T)$. Indeed, if $p \in F(T)$ then $P_T(p) = \{p\}$, hence $p \in F(P_T)$, on the other hand if $p \in F(P_T)$, since $P_T(p) \subset Tp$ we have $p \in F(T)$. By substituting T by P_T, and using a similar argument as in the proof of Theorem 3.1 we obtain the following result.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction. Suppose that $f(x, .)$ is convex and subdifferentiable on C for all $x \in C$. Let, $T : C \to P(C)$ be a multivalued mapping such that P_T is quasi-nonexpansive and satisfy the condition (E) and $S_i : C \to C, (i = 1, 2, ..., m)$, be a finite family of asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $k_n \to 1$, where $k_n = max\{k_{n,i};\, 1 \leq i \leq m\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^m F(S_i) \bigcap F(T) \bigcap Sol(f, C) \neq \varnothing$. For $C_0 = C$, let $\{x_n\}$ be sequence generated initially by an arbitrary element $x_0 \in C$ and then by

$$\begin{cases} w_n = argmin\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : \quad w \in C\}, \\ u_n = argmin\{\lambda_n f(w_n, u) + \frac{1}{2}\|u - x_n\|^2 : \quad u \in C\}, \\ y_n = \alpha_n u_n + \beta_n z_n + \gamma_{n,1} S_1^n u_n + ... + \gamma_{n,m} S_m^n u_n, \quad \forall n \geq 0, \\ C_{n+1} = \{u \in C_n : \|y_n - u\|^2 \leq \|x_n - u\|^2 + (k_n^2 - 1)\eta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases}$$

where $z_n \in P_T(u_n)$, and $\eta_n = \sup\{\|x_n - u\|^2 : u \in \mathcal{F}\} < \infty$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\} \subset [l, 1) \subset (0, 1)$, $\alpha_n + \beta_n + \sum_{i=1}^m \gamma_{n,i} = 1$ $(i = 1, 2, \cdots, m)$,

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, where $L = max\{2c_1, 2c_2\}$.

Then the sequences $\{u_n\}$ and $\{x_n\}$ converge strongly to $P_{\mathcal{F}} x_0$.

As a direct consequence of Theorem 3.1 we obtain the following convergence theorem.

Theorem 3.3. Let C be a nonempty closed convex subset of a real Hilbert space H and let F be a function from C to H such that F is monotone and L- Lipschitz continuous on C. Let, $T: C \to CB(C)$ be a quasi-nonexpansive multivalued mappings satisfying the condition (E) and $S_i : C \to C, (i = 1, 2, ..., m)$, be a finite family of asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $k_n \to 1$, where $k_n = max\{k_{n,i};\ 1 \le i \le m\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^{m} F(S_i) \bigcap F(T) \bigcap VI(F, C) \neq \varnothing$ and $T(p) = \{p\}$ for each $p \in \mathcal{F}$. For $C_0 = C$, let $\{x_n\}$ be sequence generated initially by an arbitrary element $x_0 \in C$ and then by

$$\begin{cases} w_n = P_C(x_n - \lambda_n F(x_n)), \\ u_n = P_C(x_n - \lambda_n F(w_n)), \\ y_n = \alpha_n u_n + \beta_n z_n + \gamma_{n,1} S_1^n u_n + ... + \gamma_{n,m} S_m^n u_n, \quad \forall n \ge 0, \\ C_{n+1} = \{u \in C_n : \|y_n - u\|^2 \le \|x_n - u\|^2 + (k_n^2 - 1)\eta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases}$$

where $z_n \in Tu_n$, and $\eta_n = \sup\{\|x_n - u\|^2 : u \in \mathcal{F}\} < \infty$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\} \subset [l, 1) \subset (0, 1)$, $\alpha_n + \beta_n + \sum_{i=1}^{m} \gamma_{n,i} = 1$ $(i = 1, 2, \cdots, m)$,

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$.

Then the sequences $\{u_n\}$ and $\{x_n\}$ converge strongly to $P_{\mathcal{F}} x_0$.

Proof. Putting $f(x, y) = \langle F(x), y - x\rangle$, we have that

$$argmin\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : \quad y \in C\} = P_C(x_n - \lambda_n F(x_n)).$$

Also we have

$$f(x, y) + f(y, z) - f(x, z) = \langle F(x) - F(y), y - z\rangle, \qquad x, y, z \in C.$$

Since F is a L-Lipshchitz continuous on C we get that

$$|\langle F(x) - F(y), y - z\rangle| \le L\|x - y\|\|y - z\| \le \frac{L}{2}(\|x - y\|^2 + \|y - z\|^2),$$

hence f satisfies Lischiptz-type continuous condition with $c_1 = c_2 = \frac{L}{2}$. Now, applying Theorem 3.1, we obtain the desired result. Q.E.D.

Remark 3.4. In [5], the author present a hybrid extragradient iteration method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a pseudomonotone and Lipschitz-type continuous bifunction. But in this paper we consider a shrinking projection extragradient algorithm for finding a common element of the set of common fixed points of a finite family of asymptotically nonexpansive mappings and a generalized nonexpansive set- valued mapping and the set of solutions of equilibrium problem for pseudomonotone and Lipschitz-type continuous bifunctions.

References

[1] A. Abkar, M. Eslamian, *Fixed point and convergence theorems for different classes of generalized nonexpansive mappings in CAT(0) spaces*, Comput. Math. Appl., **64** (2012) 643-650.

[2] A. Abkar, M. Eslamian, *Generalized nonexpansive multivalued mappings in strictly convex Banach spaces*, Fixed Point Theory. **14** (2013) 269-280.

[3] A. Abkar, M. Eslamian, *Geodesic metric spaces and generalized nonexpansive multivalued mappings*, Bull. Iran. Math. Soc., **39** (2013), 993-1008.

[4] P. N. Anh, *A hybrid extragradient method extended to fixed point problems and equilibrium problems*. Optimization. **62**, 271-283 (2013).

[5] P. N. Anh, *A Hybrid Extragradient Method for Pseudomonotone Equilibrium Problems and Fixed Point Problems*, Bull. Malays. Math. Sci. Soc. **36** (2013), 107-116.

[6] R. P. Agarwal, D. Oregan, D. R. Sahu, *Iterative construction of fixed point of nearly asymptotically nonexpansive mappings*, J. Nonlinear convex Anal. **8** (2007) 61-79.

[7] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994) 123-145.

[8] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image construction*, Inverse Prob. **20** (2004) 103-120.

[9] L.C.Ceng, Q.H. Ansari, J.C.Yao, *Viscosity Approximation Methods for Generalized Equilibrium Problems and Fixed Point Problems*. J. Glob. Optim. **43**, 487-502 (2009)

[10] P.L. Combettes, S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005) 117-136.

[11] S. Dhompongsa, A. Kaewkhao, B. Panyanak, *On Kirk's strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces*, Nonlinear Anal. **75** (2012) 459-468.

[12] M. Eslamian, *Hybrid method for equilibrium problems and fixed point problems of finite families of nonexpansive semigroups*, Rev. R. Accad. Cienc. Exactas Fis. Nat. Ser A. Math., **107** (2013) 299-307

[13] M. Eslamian, *General algorithms for split common fixed point problem of demicontractive mappings*. Optimization.**65** (2016), 443-465.

[14] S.D. Flam, A.S. Antipin, *Equilibrium programming using proximal-link algolithms*, Math. Program, **78** (1997) 29-41.

[15] J. Garcia-Falset, E. Llorens-Fuster, T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Math. Anal. Appl. **375** (2011) 185-195.

[16] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972) 171-174.

[17] S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974) 147-150.

[18] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953) 506-510.

[19] G. Marino, H.K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert space*, J. Math. Anal. Appl. **329** (2007) 336-346.

[20] C. Matinez-Yanes, H.K. Xu, *Strong convergence of the CQ method for fixed point processes*, Nonlinear Anal. **64** (2006) 2400-2411.

[21] A. Moudafi, *Weak convergence theorems for nonexpansive mappings and equilibrium problems*, J. Nonlinear Convex Anal., **9** (2008) 37-43.

[22] M. Mursaleen, H.M. Srivastava, Sunil K. Sharma, *Generalized statistically convergent sequences of fuzzy numbers* , J. Intelligent and Fuzzy Systems, **30** (2016) 1511-1518.

[23] K. Nakajo, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003) 372-379.

[24] J. Schu, *Weak and strong convergence of fixed points of asymptotically nonexpansive mappings*, Bull. Aust. Math. Soc. **43** (1991) 153-159.

[25] N. Shahzad, A. Udomene, *Fixed point solutions of variational inequalities for asymptotically nonexpansive mappings in Banach spaces*, Nonlinear Anal. **64** (2006) 558-567.

[26] A. Tada, W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem*, J. Optim. Theory Appl., **133** (2007) 359-370.

[27] S. Takahashi, W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331** (2007) 506-515.

[28] W.Takahashi , Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008) 276-286.